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The Ising model on a SAW

Marc Aertsens and Carlo Vanderzande

Limburgs Universitair Centrum, Universitaire Campus, 3590 Diepenbeek, Belgium

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Abstract. Using a renormalization group calculation, we show that the most probable value of the static correlation function of the Ising model on a SAW decays as a stretched exponential (with an exponent equal to the dimension D of the SAW). We prove that the thermal exponent ν_T equals D^{-1} . On the basis of extensive Monte Carlo simulations we conclude that the spin-spin autocorrelation function also decays as a stretched exponential. We find evidence for a breakdown of dynamical scaling.

1. Introduction

In this paper we study the Ising model on a 2D self-avoiding walk (SAW). This could, for example, be a model of a magnetic polymer. In particular we are interested in the regime in which the typical magnetic relaxation time τ_m is much smaller than the typical relaxation time of the polymer τ_p as given by, for example, Rouse dynamics (Rouse 1953, de Gennes 1979). In this regime the SAW can be considered as quenched. We would like to remark that the opposite limit where $\tau_m \gg \tau_p$ is also of interest as it is of importance for the so-called 'protein folding problem' (see e.g. Rammal *et al* 1986). Our main interest in the problem, will, however, be mainly from the point of view of studying the statics and dynamics of an Ising model on a fractal (Mandelbrot 1983, Stinchcombe 1985). Indeed, in recent years it has become clear that the dynamics of an Ising model on a fractal, such as the incipient infinite cluster at the percolation threshold, can be highly non-trivial (Harris and Stinchcombe 1986, Henley 1985, Jain 1986a, b, 1988). In fact one has observed such phenomena as breakdown of dynamical scaling, stretched exponential relaxation, . . . in this model. We wish to study whether such behaviour can also be found for the Ising model on a SAW, which is also a fractal.

A proper study of the dynamical aspects also requires a further understanding of the equilibrium properties of the model. So far we only know of some real space renormalization group (RG) approach to this problem (Chakrabarti and Bhattacharya 1985). One generally agrees that the Ising model on a SAW is not ordered (Stinchcombe 1983, Chakrabarti *et al* 1985), but correlation functions or the thermal exponent ν_T near zero temperature are only poorly known.

In the present paper we present several new results on both the statics and the dynamics of the problem. Our results are based on analytical reasoning and on an extensive Monte Carlo study of the model.

The paper is organized as follows. In section 2 we discuss the behaviour of the static correlation function, while in section 3 we calculate the static thermal exponent. In section 4 we discuss the dynamics of the Ising model on a fractal, which in section 5 is examined numerically for the special case of the SAW. Finally, in section 6 we present a further discussion of our results.

2. Stretched exponential decay of the correlation function

Consider an N -step SAW α on, for example, a 2D square lattice. On each site i visited by the SAW an Ising variable s_i is present. The reduced Hamiltonian of the model is then given by

$$H_\alpha = K \sum_{i,j \in \alpha} s_i s_j \tag{2.1}$$

where i and j are nearest neighbours. In this way we have also included interactions among Ising spins which are on nearest neighbour sites but which are not visited by consecutive steps of the SAW. We will call such pairs of nearest neighbours sites *bridges*. Of course, without these bridges the Ising model on a SAW would just be equivalent to the 1D Ising model, due to the fact that a SAW does not branch (figure 1).

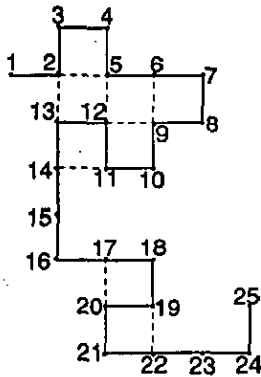


Figure 1. The full lines show a SAW of length $N = 25$. The broken lines are the bridges on this SAW.

The 2D SAW is a fractal whose fractal dimension D is $4/3$. This means that the mean Euclidean distance R between step i and step $i + M$ (with $M \gg 1$) on the SAW behaves as

$$R = A' M^{1/D} \tag{2.2}$$

where A' is a constant. This relation which is usually only considered for the end-to-end distance should also be valid here due to the self-similarity of a fractal.

It is now generally agreed that an Ising model on a fractal with $D < 2$ will only order at zero temperature $T = 0$ (Stinchcombe 1983, Chakrabarti *et al* 1985). Therefore the two-point spin correlation function G_{ij}^α on a SAW α is defined as

$$G_{ij}^\alpha = \langle s_i s_j \rangle_\alpha = \frac{\text{Tr}(s_i s_j e^{H_\alpha})}{\text{Tr}(e^{H_\alpha})} \tag{2.3}$$

We will determine an average value of G_{ij}^α for large $|i - j|$.

We therefore extend to the SAW arguments which allow us to determine the form of this correlation function and which were first given by Gefen *et al* (1983) for the case of deterministic fractals. First remark that for a SAW, G_{ij}^α is a random variable. The correlation function and thus the correlation length may then depend on the moment of G_{ij}^α which we choose to calculate (Derrida and Hilhorst 1981). For a SAW without bridges we can easily calculate the most probable value of G_{ij}^α . Indeed the

Ising model in such a case is just the 1D Ising model. For a particular SAW α we thus obtain

$$G_{ij}^\alpha \approx \exp(-N_\alpha(i, j)/\xi_1) \tag{2.4}$$

where $N_\alpha(i, j)$ is the distance between i and j along the SAW α and ξ_1 is the correlation length for the 1D Ising model which at low temperatures behaves as (Ising 1925)

$$\xi_1 \approx \xi_0 e^{2\nu_T K}$$

with $\xi_0 = 1/2$ and the thermal exponent $\nu_T = 1$. Now using (2.2) to perform an average over SAW we get

$$\overline{\ln G_{ij}} = -\frac{\overline{N_\alpha(i, j)}}{\xi_1} = -\frac{A|i-j|^D}{\xi_1} \tag{2.5}$$

where $\overline{\quad}$ denotes the average over all SAW configurations and $A = A'^{-D}$. When G_{ij} is the product of a large number of identically distributed independent random variables as is often (exactly or approximately) the case in random systems, $\exp(\overline{\ln G_{ij}})$ gives the most probable (mp) value of G_{ij} (Derrida and Hilhorst 1981). In the present case, the distribution of G_{ij} is trivially related to the distribution $P'(N_{ij})$ of N_{ij} . From the known properties of this distribution (see e.g. de Gennes 1979) we can conclude that $\overline{N_{ij}}$ differs very little from the value where $P'(N_{ij})$ reaches its maximum. For both these reasons we will refer to $\exp(\overline{\ln G_{ij}})$ as the most probable value of G_{ij} . Thus

$$G_j^{\text{mp}} \approx \exp\left(-\frac{|i-j|}{\xi_{\text{mp}}}\right)^D \tag{2.6}$$

where the most probable value of the correlation length ξ_{mp} is given by

$$\xi_{\text{mp}} = (A^{-1}\xi_1)^{1/D} \approx (A^{-1}\xi_0)^{1/D} \exp(2K/D). \tag{2.7}$$

We thus find that in the case of the SAW without bridges the most probable value of the spin-spin autocorrelation function decays as a stretched exponential. The thermal exponent in this case is $1/D = 3/4$ and thus equals the ν -exponent of the SAW itself. As pointed out by Derrida and Hilhorst (1981), G_j^{mp} corresponds to the correlation function found in Monte Carlo simulations.

What is the effect of including the bridges? It was shown by Gefen *et al* (1983) that for a deterministic non-branching fractal the value of ν_T stays at $1/D$. To obtain this result they performed a decimation-like procedure which mapped the Ising model on a fractal with bridges on to an Ising model with renormalized couplings on a fractal without bridges.

Here we will give a similar argument which will show that (2.6) and $\nu_T = 1/D$ will remain valid with bridges. Let us first define a link to be a step of the SAW which when cut breaks the set of bonds of the SAW (combined with the bridges) in two parts. We will denote the number of links in a SAW α by L_α . This number was recently studied numerically by Seno and Stella (1989) who found the average number of links \bar{L} :

$$\bar{L} = \frac{1}{2}N(1 + O(N^{-\Delta})). \tag{2.8}$$

We will proceed to show that any N -step SAW α can be mapped through a decimation-like procedure on to a SAW of L_α steps, where, however, the Ising couplings have been

renormalized. Our procedure goes as follows. We move along the SAW until we find a bond that is not a link; let this be the bond $k \rightarrow k+1$. Let the next bond which is again a link be $l \rightarrow l+1$ (in the example of figure 1 $k=2$ and $l=14$). We will call the set of spins s_{k+1}, \dots, s_{l-1} and the bonds involving at least one of them a blob. Consequently we trace out all the spins inside the blob. This can be done as follows:

$$\exp(K's_k s_l + C') = \sum_{s_{k+1}=\pm 1} \dots \sum_{s_{l-1}=\pm 1} \exp(H_\alpha(s_{k+1}, \dots, s_{l-1})) \tag{2.9}$$

where C' is a constant and $H_\alpha(s_{k+1}, \dots, s_{l-1})$ includes all interactions from (2.1) involving the set of spins s_{k+1}, \dots, s_{l-1} . In general K' will be a complicated function of K which depends on the structure of the SAW between sites k and l . Of course, in practice it may be very difficult to calculate K' . We now proceed along the SAW until we encounter the next blob. The spins in this blob are traced out, etc. . . . When we have reached the end of the SAW, we will have obtained a SAW with only links, or equivalently just a 1D chain. The interactions between Ising spins on this chain will be K for all bonds which were links in the original SAW, but will take on a different value, depending on the original structure of the SAW, on the other bonds. Figure 2 shows the result of applying our decimation procedure to the SAW of figure 1.

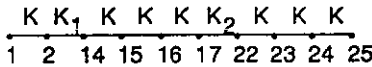


Figure 2. SAW of figure 1 on which the decimation procedure described in the text has been applied. For simplicity the SAW has been drawn as a line of renormalized bonds.

Considering the set of all N -step SAWs (with N large), the statistics of the Ising model on a SAW can thus be mapped on to that of a 1D Ising model with random couplings. The geometrical randomness of the SAW is transformed into the randomness of the Ising couplings. These couplings will be K with a probability $p > 0.5$ (due to the result (2.8)). When they are different from K , they can take on values K' given by a distribution $P(K')$. This is an essential difference between the present case and that of a deterministic fractal where all renormalized couplings are the same.

The length of the chain obtained by 'decimating' a SAW in the way described above will depend on the configuration of the SAW. Notice, however, that each bond with $K' \neq K$ has at least one neighbour with Ising coupling K . Let, for example, the bond $k \rightarrow k+1$ be given by K_1 , and the bond $k+1 \rightarrow k+2$ by K . Then we can define the following renormalization mapping

$$\exp(K''s_k s_{k+2} + C'') = \sum_{s_{k+1}=\pm 1} \exp((K_1 s_k + K s_{k+2})s_{k+1}) \tag{2.10}$$

where C'' is again a constant. We find

$$K'' = \frac{1}{2} \ln \left(\frac{\cosh(K_1 + K)}{\cosh(K_1 - K)} \right). \tag{2.11}$$

After performing this renormalization for all bonds with $K' \neq K$, our chain will have just as many bonds as the original SAW α had links, i.e. L_α . On this new chain, Ising interactions will have strength K with probability q and strength $K'' \neq K$ with probability $Q(K'')$ where q and Q can be calculated in principle using (2.11), p and $P(K')$. We remark that the mapping (2.9) and (2.10) is an exact one.

On the basis of the result (2.8) we can also conclude that on average our renormalized SAW will have half of the bonds of the original SAW and that therefore the average rescaling factor is $2^{1/D}$.

We will now use the transformation of a correlation function under a RG-mapping to determine $\overline{\ln G_{ij}}$. The transformation law for an arbitrary correlation function G_{ij} , not necessarily that of the present problem, is given by

$$G_{ij} = G(|i-j|, u) \approx b^{-2x} G' \left(\frac{|i-j|}{b}, b^{1/\nu_\tau} u \right) \tag{2.12}$$

where we have assumed spatial isotropy and have explicitly denoted the dependence on the thermal scaling field u . In our case, for a transition at zero temperature, we have $u = \exp(-2K)$. We assume that (2.12) holds in particular for the most probable value $\exp(\overline{\ln G_{ij}})$ of the correlation function in our model.

The correlation function G'_{ij} in the renormalized SAW system can be calculated in a straightforward way. It is still given by (2.4), where now, however, ξ will be a random variable depending on the distribution of Ising interactions within the particular chain. On average we thus have

$$\overline{\ln G'_{ij}} \approx - \overline{N'(i, j) / \xi} \approx - \overline{N'(i, j) / \bar{\xi}} \tag{2.13}$$

(assuming independence between length and correlation length of the renormalized chain, which of course is not completely valid; see also the end of section 3), where $N'(i, j)$ is the number of bonds between i and j in the renormalized system, and $\bar{\xi}$ is given by (Derrida and Hilhorst 1981)

$$\bar{\xi} = \left[q |\ln \tanh K| + (1-q) \int dK'' Q(K'') |\ln \tanh K''| \right]^{-1}. \tag{2.14}$$

Now, using (2.8) and (2.2) in (2.13) we can write

$$\overline{\ln G'_{ij}} \approx -0.5 \overline{N'(i, j) / \bar{\xi}} \approx -0.5 A |i-j|^D / \bar{\xi} = -(|i-j| / \xi_{mp})^D \tag{2.15}$$

with

$$\xi_{mp} = (2A^{-1} \bar{\xi})^{1/D}. \tag{2.16}$$

Now we expect that ξ_{mp} has the following form:

$$\xi_{mp} \sim \exp(2\nu_\tau K) \sim u^{-\nu_\tau} \tag{2.17}$$

which, using (2.12), finally gives

$$\begin{aligned} \overline{\ln G_{ij}} &\approx -2x \overline{\ln b} + \overline{\ln G' \left(\frac{|i-j|}{b}, b^{1/\nu_\tau} u \right)} \\ &\approx -2x \overline{\ln b} - (|i-j| / \xi_{mp})^D. \end{aligned} \tag{2.18}$$

In this way we find that, as for the SAW without bridges,

$$G_j^{mp} \approx \exp \left(- \frac{|i-j|}{\xi_{mp}} \right)^D \tag{2.19}$$

i.e. the correlation function decays as a stretched exponential. This is a remarkable result. It could be of quite some interest to investigate whether such a decay also occurs for the Ising model on other random fractals or whether it is a result of the essentially linear character of the SAW.

3. The thermal exponent ν_T

We will now proceed to show that for $K \rightarrow \infty$ the decimation (2.9) can be worked out, leading to $\nu_T = 1/D$. Therefore, consider first the case that in (2.9) $s_k = s_l$. Then for $K \rightarrow \infty$, the right-hand side of (2.9) will be dominated by that configuration where all spins in the blob are equal to s_k . Let the number of bonds in the blob be m . In that case (2.9) reads

$$\exp(K' + C') = \exp(mK)[1 + O(\exp(-K))]. \quad (3.1)$$

When $s_k = -s_l$ and $K \rightarrow \infty$, the right-hand side of (2.9) is dominated by those configurations in which one interface of minimal length is present inside the blob. Denote by n_0 the number of bonds broken in such an interface, and by g the degeneracy of such a configuration (on any 2D lattice $n_0 \geq 2$). We get

$$\exp(-K' + C') = g \exp[(m - 2n_0)K][1 + O(\exp(-K))]. \quad (3.2)$$

Taking (3.1) and (3.2) together we get to leading order

$$\exp(2K') \approx g^{-1} \exp(2n_0K). \quad (3.3)$$

Therefore asymptotically ($K \rightarrow \infty$), $K' \approx K$. Using this result in (2.11) and considering again the limit $K \rightarrow \infty$, we get $K'' = K$ asymptotically. So at very low temperatures, our chain will from a coarse-grained point of view, look like a 1D chain with all degrees of freedom inside the blob frozen in. The randomness thus disappears out of the chain and from (2.14) we then get $\bar{\xi} = \frac{1}{2} \exp(2K)$. Then, trivially, from (2.16) and (2.17), it follows that $\nu_T = 1/D$, as in the case of the SAW without bridges. This can be further understood by noticing that at very low temperatures the blob spins are frozen in, and only the 1D set of spins on links determines the behaviour.

Thus as in the case of percolation, the statics are determined by the links. We will now proceed to discuss the dynamics of the chain where the reverse is true; it is determined by the blobs. Finally we would like to remark that the results of the present section indicate that for $K \rightarrow \infty$, the second equality of (2.13) involves no approximation because at these low temperatures ξ is no longer random. Thus the stretched exponential decay becomes exact in that limit.

4. Glauber dynamics of Ising model on a fractal

We now turn to dynamics. The dynamics we will be interested in is Glauber dynamics (model A, Hohenberg and Halperin 1977). The case of an Ising model on a SAW without bridges can easily be solved as it is equivalent to the dynamics of the 1D Ising model which was solved in an important paper by Glauber (1963). The spin-spin time-dependent correlation function $G_{ij}(t)$ and the autocorrelation function $C(t)$ which is defined as

$$C(t) = \frac{1}{N} \sum_{i=1}^N \langle s_i(0)s_i(t) \rangle \quad (4.1)$$

were found to decay exponentially with a relaxation time $\tau \sim (1 - \tanh 2K)^{-1}$. For $K \rightarrow \infty$ we obtain

$$\tau \sim e^{4K}. \quad (4.2)$$

This result should also hold for the SAW without bridges. In general the dynamic exponent z is related to the correlation length ξ via

$$\tau \sim \xi^z. \tag{4.3}$$

Combining (2.7) and (4.2) leads, for the SAW without bridges, to the result

$$z = 2D = \frac{8}{3}. \tag{4.4}$$

For the 1D Ising model one would have $z = 2$. The difference is due to the stretched exponential decay of the static correlation functions for the Ising model on a SAW.

When adding the bridges we can expect all kinds of effects. First of all it is known that for Ising models on such fractals as the Sierpinski gasket or the incipient infinite cluster at percolation, the scaling law (4.3) breaks down (Harris and Stinchcombe 1986, Stinchcombe 1985, Henley 1985, Jain 1986a, b, Nunes de Silva and Lage 1987). To be more precise, z becomes a function of temperature:

$$z = z_0 K + z_1. \tag{4.5}$$

Secondly, the autocorrelation function $C(t)$ was found by Jain (1988) to decrease as a stretched exponential for the case of an Ising model on percolation clusters. Such stretched exponential decay seems to be ubiquitous, not only in systems with quenched randomness (such as spin glasses or random field models) but even in the low-temperature region of the Ising model (Huse and Fisher 1987). In general Lifshitz arguments (Lifshitz 1968) are used to predict such stretched exponential decays (see e.g. Bray 1988). Numerical work, however, leads to exponents different from those predicted by simple arguments (see Jain 1988). The Ising model on a SAW has both random and fractal aspects, so we may also expect such decay here. Remark also that on a SAW with bridges both $C(t)$ and τ again become random variables. In discussions in the literature it is very seldom stated what is meant by 'the relaxation time'. As most results come from numerical simulations we will always refer to the most probable value τ_{mp} which we will take to be related through equation (4.3) with the correlation length ξ_{mp} .

Let us discuss in more detail the arguments leading to (4.5) to see whether they are also valid for the SAW. For a system which only orders at zero temperature, at low temperatures the dynamics are determined by domain-wall diffusion (Cordery *et al* 1981). On a fractal this diffusion is slowed down when the domain wall has to cross a blob. Consider again a SAW α . Let $n^\alpha(b)$ be the maximum length of the interface during a realization of the diffusion process across a blob b , then we can expect the diffusion along such a blob to take a time $\tau^\alpha(b)$ given by

$$\tau^\alpha(b) \approx \tau_0 \exp[2Kn^\alpha(b)] \tag{4.6}$$

where τ_0 is, for example, the time to cross one link.

In a given SAW α of size (e.g. end-to-end distance) R there will be several different relaxation times $\tau^\alpha(b)$. Following Henley (1985) we define τ_{mp} as the typical (or most probable) value of the maximum relaxation time, i.e.

$$\tau_{mp} = \exp(\overline{\ln \tau_{max}^\alpha}) \tag{4.7}$$

with

$$\tau_{max}^\alpha = \max_b \tau^\alpha(b) \tag{4.8}$$

where the maximum is taken over all blobs in the SAW α . Now, of course, using (4.6) and the monotonicity of the ln-function

$$\overline{\ln \tau_{max}^\alpha} = \ln \tau_0 + 2K \overline{\max n(b)}. \tag{4.9}$$

If we increase the size R of the system, it can be shown (Henley 1985) that due to the self-similar structure of the fractal, the maximum value of $n(b)$ will increase logarithmically with R . Thus we have

$$\overline{\max n(b)} \approx z' \ln R + z'' \quad (R \rightarrow \infty) \quad (4.10)$$

where z' and z'' are constants. This result should hold on any finitely ramified fractal and should therefore also hold for the SAW. Taking a sequence of SAWs with size $R = \xi_{mp} \sim \exp(2\nu_T K)$ as $K \rightarrow \infty$ we get from (2.7) and (4.7)–(4.10)

$$\ln \tau_{mp} = \ln \xi_{mp}(z_0 K + z_1) + C' \quad (4.11)$$

where C' is a constant, $z_0 = 2z'$ and $z_1 = z''/\nu_T$. This is the result (4.5).

Thus, on a certain SAW α , both the autocorrelation function $C(t)$ and the time τ_{mp} depend on the distribution of blobs within this SAW. As we are unable to say anything about the average value of these quantities analytically we have resorted to a numerical calculation.

5. Numerical results

In our numerical work we have proceeded as follows. First we have generated a set of SAW of N steps (we did our calculations for $N = 26, 101$ and 201). We then studied for each SAW the dynamics of the Ising model on that SAW, using a standard metropolis algorithm. After waiting for a suitable long equilibration time, we calculated the autocorrelation function $C(t)$. This quantity was then averaged over the set of SAWs. The averaged quantity gives a good estimate of the most probable value $C^{mp}(t)$ of this autocorrelation function. We have investigated this quantity as a function of temperature up to the relatively low temperature $K = 2$ (for the Ising model on percolation a similar study investigated only the region $K \leq 1.5$ (Jain 1986)). As the relaxation times at these low temperatures grow to large values, the number of SAWs which could be studied at those temperatures was rather small (≈ 40 SAWs of length 201). At higher temperatures much larger numbers of SAWs were investigated (up to 3000 SAWs). In order to compensate for the small number of SAWs at low temperatures, we have used a technique that takes a representative sample of SAWs. This technique is discussed in the appendix.

Our results are the following. We find very clear evidence for a stretched exponential decay of the autocorrelation function

$$C^{mp}(t) \approx e^{-(t/\tau)^\beta}. \quad (5.1)$$

In figure 3 we present some typical results. We plot $\ln(-\ln C^{mp}(t))$ versus $\ln t$. After a short cross-over time we see that the data fit the form (5.1) very well. From a fit of the data we can determine β and τ . In figure 4 we plot our results for β as a function of K . As in the case of the Ising model on percolation (Jain 1988), the exponent β is temperature dependent. It may be possible that at low temperatures the value of β reaches a constant value of $\beta \approx 0.25$.

We now turn our attention to the temperature dependence of τ . Combining (4.3), (4.5) and (2.17) we get

$$\ln \tau \approx AK^2 + BK + C \quad (5.2)$$

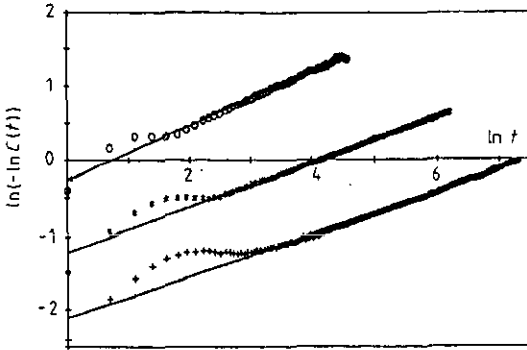


Figure 3. Stretched exponential decay of the autocorrelation function $C^{mp}(t)$ for $K = 0.80$ (top), $K = 1.25$ (middle) and $K = 1.55$ (bottom). The fitted values for the slopes β are $\beta = 0.40$, $\beta = 0.30$ and $\beta = 0.27$ respectively.

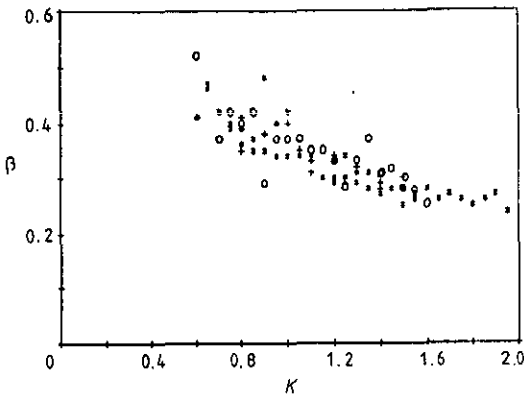


Figure 4. The exponent β as a function of K . We have used the symbols $+$, $*$ and \circ for SAW lengths $N = 26$, 101 and 201 respectively.

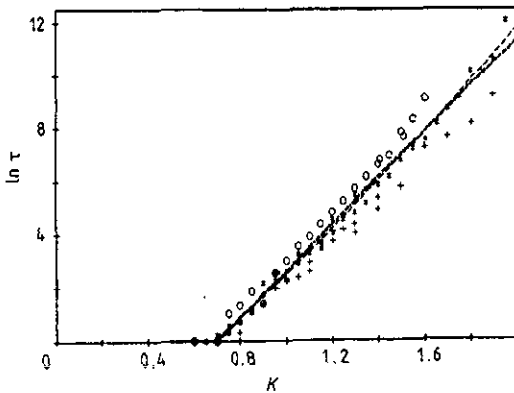


Figure 5. Plot of $\ln \tau$ against K . The symbols have the same meaning as in figure 4. The fitted line has slope $B = 8.7$ and ordinate $C = -6.1$. The broken line is the parabola with $A = 1.1$, $B = 6.1$ and $C = -4.7$ (see table 1).

with C a constant and

$$A = 2\nu_T z_0 \quad B = z_0 \ln \xi_0 + 2\nu_T z_1. \quad (5.3)$$

Dynamical scaling would imply $A = z_0 = 0$ and $B = 2\nu_T z$. Figure 5 shows our data for $\ln \tau$ versus K . If they are fitted to the form (5.2) with $A = 0$ (i.e. assuming dynamical scaling) we get the results shown on the left in table 1. If we use the value $B \approx 8.7$ (± 0.2) (coming from taking our data for all values of N), $\nu_T = 0.75$ implies $z \approx 5.8$, a value which seems unusually large and indicates a possible breakdown of scaling.

Table 1. Fitting results for $\ln \tau$ to the formulae (5.2). The numbers B and C on the left-hand side of the table are fitted with $A = 0$.

B	C	SAW length	A	B	C
9.1 ± 0.2	-5.4 ± 0.4	$N = 26$	0.1 ± 1.0	7.6 ± 0.1	-5.5 ± 0.6
8.8 ± 0.2	-6.2 ± 0.2	$N = 101$	1.3 ± 0.7	5.5 ± 0.5	-4.3 ± 0.4
9.2 ± 0.2	-6.1 ± 0.4	$N = 201$	2.0 ± 1.5	4.6 ± 2.3	-3.7 ± 0.8
8.7 ± 0.2	-6.1 ± 0.2	all N	1.1 ± 1.1	6.1 ± 1.2	-4.7 ± 0.6

If we let the parameter A in (5.2) free, we get the best fits of A , B and C shown on the right of table 1. We see that the coefficient A is small and that the value zero cannot be completely ruled out. The value of A , however, also increases with increasing system size. On the basis of these results we are led to conclude that there is a weak breakdown of scaling in the Ising model on a SAW. This is indeed what would be predicted for any fractal from the arguments in section 4.

6. Conclusions

In the present paper we have investigated the properties of the Ising model on a SAW. We have found that both the static correlation function and the autocorrelation function decay as a stretched exponential. In the latter case this conclusion is based only on numerical evidence, but in the former case it comes from analytical reasoning which in the zero temperature limit becomes exact. Our result is to the best of our knowledge the first such exact result for a random fractal. This result could be achieved thanks to the essentially linear structure of the SAW, which completely determines its static behaviour. We could also show, using analytical arguments based on a renormalization mapping, that the thermal exponent ν_T describing the most probable value of the correlation function, equals $1/D$. Of course this does not rule out the possibility that the average (or other moments of) correlation length needs another thermal exponent.

We have also found numerical evidence for a breakdown of dynamical scaling. The interesting dynamical properties are mainly caused by the blobs occurring within the SAW.

Other interesting phenomena may occur when we include a magnetic field. In that case our renormalization transformation maps the Ising model on a SAW into a 1D chain with random interactions and random fields. The dynamics of such a chain can be related to Sinai-diffusion (Stella and Vanderzande 1991) and can therefore be of a completely new type for spin systems.

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Appendix: Generation of the SAWs

We wish to calculate the value of some physical quantity A , averaged over all self-avoiding random walks (SAW) of length N . If c_N is the number of all SAWs of length N , it is straightforward to choose at random M SAWs out of the c_N SAWs. Then one calculates the quantity A for each SAW α and we call the result A_α ($\alpha = 1, \dots, M$). The average value $\langle A \rangle$ is then estimated by

$$\langle A \rangle \approx \frac{1}{N} \sum_{\alpha=1}^M A_\alpha. \quad (\text{A.1})$$

The probability of choosing a certain SAW α ($\alpha = 1, \dots, c_N$) is $p_\alpha = 1/c_N$. The problem is how to choose the SAWs at random if you do not have a list of all c_N SAWs of length N .

A solution is to generate SAWs according to the method proposed by Rosenbluth and Rosenbluth (1955) or its extension (Meirovitch 1983). Using this method, some SAWs are generated more than others. A weighting function W_α is introduced such that all configurations are counted equally. Thus if r_α is the probability that SAW α is generated by the Rosenbluth method,

$$\frac{1}{c_N} = p_\alpha = C W_\alpha r_\alpha \quad (\text{A.2})$$

where C is a constant.

Generating M SAWs according to the Rosenbluth method, the average value $\langle A \rangle$ is approximated by

$$\langle A \rangle \approx \frac{\sum_{\alpha=1}^M W_\alpha A_\alpha}{\sum_{\alpha=1}^M W_\alpha}. \quad (\text{A.3})$$

Estimating $\langle A \rangle$ this way is referred to in this appendix as (I). This method is not very effective if the time to calculate A_α is one or more orders of magnitude larger than the time to generate the SAW α . Indeed, according to the definition of r_α , the Rosenbluth method generates mostly SAWs with large r_α . According to (A.2), the weight factor W_α of such SAWs is small. Thus the contribution of $W_\alpha A_\alpha$ to the value $\langle A \rangle$ is also relatively small. Especially for large N , when the weight factors differ by several orders of magnitude, it remains possible that for instance the fiftieth generated SAW has a higher weight W_{50} than the sum of the 49 previous values of W_α (see figure 6). Hence in order to have a reliable estimate of $\langle A \rangle$, one needs to compute A_α on a large number of SAWs α .

In order to have a reliable estimate of $\langle A \rangle$ by calculating A_α for a relatively low number of SAWs, we first generated a large number of SAWs. These SAWs were distributed in classes according to their weight W_α . The quantity A_α was calculated only for some SAWs. These were selected in such a way that the measured distribution of weights W_α was always approximated as closely as possible.

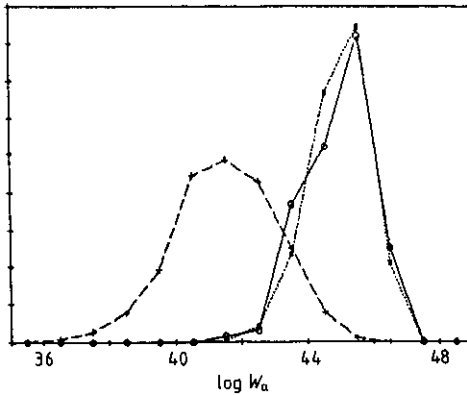


Figure 6. The Rosenbluth method is used to generate 300 000 SAWs of length $N = 101$. The SAWs are distributed in classes, such that SAWs with the same integer for $\log W_i$ belong to the same class. The broken curve gives the number of SAWs contained in every class. The sum of the weights W_i of all the SAWs in a class is given by the broken curve. One remarks that although few SAWs are generated in the classes with large $\log W_i$ these classes contain the majority of the total weight. If the SAWs are selected according to (II), only 120 SAWs are accepted. These are distributed according to the full curve. The coincidence of the dotted and the full curve confirm formulae (A.2) and justify method (II). For the acceptance/rejection of the SAWs, we have used $W'_{\max} = 0.1 \times 10^{48}$ ($W_{\max} = 4 \times 3^{99} \approx 0.7 \times 10^{48}$).

In order also to avoid the problem that most generated SAWs have a low weight W_α , we used an acceptance-rejection procedure:

- (i) Generate a SAW α according to the Rosenbluth procedure and calculate its weight W_α .
- (ii) Generate a random number R ($0 \leq R \leq 1$) and call W_{\max} the maximum value of W_α over all c_N SAWs of length N .
- (iii) (a) If $R \leq W_\alpha / W_{\max}$: the SAW is accepted and A_α is calculated. (b) Else if $R \geq W_\alpha / W_{\max}$: the SAW is not accepted.

This procedure is repeated for a sufficiently large number of SAWs. Because SAWs are generated according to a density function r_α , taking into account (A.2), the accepted SAWs have on average the density function p_α . Thus if N SAWs are accepted, the value $\langle A \rangle$ is estimated by (A.3). This method is referred to as (II).

It is obvious that in (II) more computer time is needed to generate an accepted SAW than in (I). So if the time to calculate A_α is smaller than the time to generate SAW α , method (I) seems the best choice.

For large c_N the probability of choosing a SAW with a weight near W_{\max} is very small. One has to generate a very large number of SAWs in order to have one accepted. This is avoided if we replace in (II) W_{\max} by a smaller number W'_{\max} . If all the generated SAWs have a weight $W_\alpha \leq W'_{\max}$, the accepted SAWs also have in mean the density function p_α and the computer time used to generate the same number of SAWs is reduced by the factor (W'_{\max} / W_{\max}) . If, however, a SAW with a weight factor $W_\alpha > W'_{\max}$ is generated, the calculations must be restarted.

For very large N , most generated SAWs are rejected and thus method (II) is not very effective either. A good alternative might be to use the Beretti-Sokal method (Beretti and Sokal 1985) or the Pivot algorithm (Madras and Sokal 1988, Lal 1969, Kremer and Binder 1988).

We generated saws of length $N = 26, 101$ and 201 . For $N = 201$ the calculations at different temperatures were executed on the same 44 saws, generated according to method (II). For $N = 26$ and $N = 101$ we generated the saws during the calculations. Both methods (I) and (II) are used for $N = 26$ and $N = 101$.

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